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The q-deformed binomial distribution and its asymptotic behaviour

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Abstract. The q-analogue of the binomial distribution is defined by virtue of the q-binomial theorem, which takes the Euler distribution as its limiting form and is new to the literature.

1. Introduction

The binomial counting probability distribution $P_B(k; n, \tau)$ and the Poisson distribution $P_P(k, \lambda)$ are well-studied discrete distributions of the random variable k, that are extensively used in various areas of sciences. For instance, in quantum optics, the coherent state and the binomial state are linear combinations of the numer states with coefficients chosen such that the photon-counting distributions are Poissonian or binomial, respectively [1, 2]. Since the Poisson distribution $P_P(k, \lambda)$ can be used as a convenient approximation to the binomial distribution $P_B(k; n, \tau)$ when n is large and τ is small (but $n\tau$ is constant λ) (the so-called Poisson's theorem), the binomial state may reduce to the coherent state and to the number state in different limits.

Recently much attention has been paid to q-deformed oscillators [3]. The single-mode q-Heisenberg algebra is defined as [4]

$$aa^+ - qa^+a = 1 \qquad [N, a^+] = a^+ \qquad [N, a] = -a. \tag{1}$$

The q-analogue of the number state

$$|n\rangle = \frac{(a^+)^n}{\sqrt{[n]_q!}}|o\rangle \qquad [n]_q! = [n]_q[n-1]_q \ldots [1]_q \qquad [n]_q = \frac{1-q^n}{1-q} \tag{2}$$

and the q-analogue of the coherent state

$$|\alpha\rangle = (\exp_q(|\alpha|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q!}}|n\rangle \qquad e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{3}$$

have been extensively studied by many others. The density matrix for a pure q-coherent state is

$$\rho = |\alpha\rangle\langle\alpha|. \tag{4}$$

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The q-deformed Poisson distribution can be derived by averaging the density matrix in a q-number state

$$\langle n|\rho|n\rangle = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{[n]_q!}(\exp_q(|\alpha|^2))^{-1}. \tag{5}$$

Thus an interesting question arises naturally, that is how to define a q-analogue of the binomial state so that it can reduce to the q-number state and to the q-coherent state in different limits. To answer this question, we need, mathematically, a q-deformed binomial distribution which takes the q-Poisson distribution as its limiting form and reduces to the ordinary binomial distribution when $q \to 1$.

In mathematical literature, the q-deformed Poisson distribution is known as the Euler distribution. In addition, there is another kind of q-analogue of the Poisson distribution which is called the Heine distribution. These were put forward by Benkherouf and Bather [5] as feasible prior distributions in their study of stopping time strategies when sequentially drilling for oil. Kemp [6] has shown that the Euler and Heine distributions are limiting forms of a q-analogue of the Pascal (or negative binomial) distribution and a q-analogue of the binomial distribution, respectively. However, in order to build the binomial state for the q-boson case, a new kind of q-analogue of the binomial distribution is appealing, which takes the Euler, but not Heine, distribution as its limiting form.

This paper is devoted to defining a new q-analogue of the binomial distribution by virtue of the q-deformed binomial theorem and studying its limiting form as well as its properties, e.g. moments and recurrence relationships. This paper is arranged as follows: in section 2, we recall some facts about the Euler and Heine distributions in terms of our notation system; in section 3, with the aid of the q-binomial theorem, we introduce the new q-deformed binomial distribution and prove its limiting form is the Euler; in section 4, some properties and applications of the new q-deformed binominal distribution are discussed.

2. The Euler and Heine distributions

Throughout this paper, we use the following notation

$$(x+y)_q^n = \prod_{k=0}^{n-1}(x+q^k y) \qquad e_q^x = \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!} \qquad e_{1/q}^x = \sum_{n=0}^{\infty}\frac{x^n}{[n]_{1/q}!} \tag{6}$$

where $[n]_q = q^{n-1}[n]_{1/q}$, so that all expressions can reduce to the familiar non-deformed ones in the limit $q \to 1$. From Jackson [7], we have the following relations:

$$(x+y)_q^{n+m} = (x+y)_q^n (x+q^n y)_q^m \qquad \lim_{n\to\infty}(1-x)_q^n = (e_q^{x/1-q})^{-1} \qquad (e_q^x)^{-1} = e_{1/q}^{-x}. \tag{7}$$

As $q \to 1$ and if also $x \to 0$ such that $x/(1-q) = \lambda$ remains finite, the right-hand side of the above limiting formula tends to $e^{-\lambda}$. Also from (6) and (7), the Euler identity reads

$$S(x,q) = \prod_{n=0}^{\infty}(1-xq^n)^{-1} = e_q^{x/1-q} = \sum_{n=0}^{\infty}\frac{x^n}{(1-q)_q^n} = 1 + \frac{x}{1-q} + \frac{x^2}{(1-q)(1-q^2)} + \ldots. \tag{8}$$

Similarly, another identity is also obtained

$$\prod_{n=0}^{\infty}(1+xq^n) = e_{1/q}^{x/1-q} = \sum_{n=0}^{\infty}\frac{x^n q^{n(n-1)/2}}{(1-q)_q^n}. \tag{9}$$

Using the Euler identity (8), one can express the probability mass function (PMF) of the Euler distribution as

$$P_{\mathrm{E}}(k;\alpha,q)=\frac{\alpha^k}{(1-q)_q^k}S(\alpha,q)^{-1}=\frac{(\alpha/1-q)^k}{[k]_q!}(\mathrm{e}_q^{\alpha/1-q})^{-1}$$
$$0<q<1 \qquad 0<\alpha<1. \tag{10}$$

This is the q-deformed Poisson distribution with parameter $\alpha/(1-q)$ exactly. From (9), the Heine distribution has PMF

$$P_{\mathrm{H}}(k;\beta,Q)=\frac{(\beta/1-Q)^k}{[k]_{1/Q}!}(\mathrm{e}_{1/Q}^{\beta/1-Q})^{-1} \qquad 0<Q<1 \qquad 0<\beta. \tag{11}$$

Their probability generating functions (PGFs) are

$$G_{\mathrm{E}}(z,\alpha,q)=\sum_{k=0}^{\infty}P_{\mathrm{E}}(k;\alpha,q)z^k=\mathrm{e}_q^{\alpha z/1-q}(\mathrm{e}_q^{\alpha/1-q})^{-1} \tag{12}$$

$$G_{\mathrm{H}}(z,\beta,Q)=\sum_{k=0}^{\infty}P_{\mathrm{H}}(k;\beta,Q)z^k=\mathrm{e}_{1/Q}^{\beta z/1-Q}(\mathrm{e}_{1/Q}^{\beta/1-Q})^{-1} \tag{13}$$

respectively. Thus it is easy to find out the interrelationship between the Euler and Heine distributions, i.e. their PMFS can be written in terms of one formula

$$P(k;\alpha,q)=\frac{(\alpha/1-q)^k}{[k]_q!}(\mathrm{e}_q^{\alpha/1-q})^{-1} \tag{14}$$

where $0<q<1$, $0<\alpha<1$ for the Euler, and $1<q$, $\alpha<0$ for the Heine distribution. In fact, replacing q by Q^{-1} and α by $-\beta Q^{-1}$ in (10) and (12), we obtain (11) and (13), respectively. From (10) and (11) we see clearly that the transitional distribution between Euler and Heine is Poissonian. Therefore, both Euler and Heine distributions can be recarded as the q-deformed Poisson distribution. Kemp introduced a third member of this q-deformed family of the Poisson distribution, the pseudo-Euler distribution with PMF and PGF[6]

$$P_{\mathrm{pE}}(k;\alpha,u)=\frac{(\alpha/1+u)^k}{[k]_{-u}!}(\exp_{u^2}(\alpha/1-u^2))^{-1}\exp_{u^2}(-\alpha u/1-u^2)$$
$$0<u<1 \qquad 0<\alpha<1. \tag{15}$$

$$G_{\mathrm{pE}}(z,\alpha,u)=\frac{\exp_{u^2}(\alpha z/1-u^2)\exp_{u^2}(-\alpha u/1-u^2)}{\exp_{u^2}(-\alpha zu/1-u^2)\exp_{u^2}(\alpha/1-u^2)} \tag{16}$$

respectively, where

$$[k]_{-u}=\frac{1-(-u)^k}{1-(-u)}.$$

Comparing (15) with (14), we find that replacing q by $-u$ in (14) leads to (15) exactly. To summarize, the Euler, Heine and pseudo-Euler distributions arise from (14) when $0<q<1$, $1<q$ and $-1<q<0$, respectively. Also based on the fact

$$0<P^2(1;\alpha,q)/P(0;\alpha,q)P(2;\alpha,q)=1+q \tag{17}$$

there is no other member in the q-family of the Poisson distribution, i.e. no valid distribution is possible when $q<-1$.

3. The q-deformed binomial distribution

The Euler PGF $G_E(z, \alpha, q)$ can be expressed as a limiting form of PGF

$$G_{NB}(n; z, \alpha, q) = (1-\alpha)_q^n/(1-\alpha z)_q^n \qquad G_E(z, \alpha, q) = \lim_{n\to\infty} G_{NB}(n; z, \alpha, q). \tag{18}$$

Using the q-deformed Newton binomial theorem [8]

$$((1-z)_q^n)^{-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q z^k \qquad 0<q<1 \qquad |z|<1 \qquad n=1,2,3,\ldots \tag{19}$$

one can define a q-analogue of the Pascal (i.e. negative binomial) distribution with PMF

$$P_{NB}(k; n, \alpha, q) = \binom{n+k-1}{k}_q \alpha^k (1-\alpha)_q^n$$
$$0<q<1 \qquad 0<\alpha<1 \qquad k=0,1,2,\ldots \tag{20}$$

from (19), where

$$\binom{n}{k}_q = [n]_q!/[k]_q![n-k]_q!.$$

When $q\to 1$, (20) reduces to the standard Pascal distribution. The q-analogue of the negative binomial distribution is also known as the inverse absorption distribution, first studied by Dunkl in 1981 [9]. Of course, when $n\to\infty$, the q-deformed negative binomial distribution tends to the Euler PMF

$$\lim_{n\to\infty} P_{NB}(k; n, \alpha, q) = \lim_{n\to\infty} \frac{[n+k-1]_q[n+k-2]_q\ldots[n]_q}{[k]_q!}\alpha^k(1-\alpha)_q^n$$
$$= \frac{(\alpha/1-q)^k}{[k]_q!}(e_q^{\alpha/1-q})^{-1} = P_E(k; \alpha, q) \tag{21}$$

where the fact

$$\lim_{n\to\infty} [n]_q = \frac{1}{1-q}$$

has been used.

Similarly, the Heine PGF $G_H(z, \beta, Q)$ can be expressed as a limiting form of PGF:

$$G_B^*(n; z, \beta, Q) = (1+\beta z)_Q^n/(1+\beta)_Q^n \qquad G_H(z, B, Q) = \lim_{n\to\infty} G_B^*(n; z, \beta, Q). \tag{22}$$

The distribution with PGF (22) was discussed by Kemp and Kemp in 1991 [10]. With the aid of the q-binomial theorem [11,7]

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} x^{n-k} y^k \qquad n=1,2,3,\ldots \tag{23}$$

we have the PMF of the distribution (22) as

$$P_B^*(k; n, \beta, Q) = \binom{n}{k}_Q Q^{k(k-1)/2}\beta^k((1+\beta)_Q^n)^{-1} \qquad 0<Q<1 \quad 0<\beta \quad k=0,1,\ldots,n. \tag{24}$$

When $n\to\infty$, (24) tends to the Heine PMF

$$\lim_{n\to\infty} P_{\mathrm{B}}^{*}(k;n,\beta,Q)=\lim_{n\to\infty}\frac{[n]_Q[n-1]_Q\ldots[n-k+1]_Q}{[k]_Q!}Q^{k(k-1)/2}\beta^k((1+\beta)_Q^n)^{-1}$$

$$=\frac{(\beta/1-Q)^k}{[k]_{1/Q}!}(\mathrm{e}_{1/Q}^{\beta/1-Q})^{-1}=P_{\mathrm{H}}(k;\beta,Q). \tag{25}$$

So far we have two kinds of distribution (i.e. P_{NB} and P_{B}^{*}), of which the limiting forms are the Euler and Heine distributions, respectively. Noticing that the q-Poisson distribution (5) is identical with the Euler distribution, to define a reasonable binomial state for the q-boson, we have to introduce another kind of q-binomial distribution which takes the Euler distribution as its limiting form and is new to the literature. We define the q-deformed binomial distribution as

$$P_{\mathrm{B}}(k;n,\tau,q)=\binom{n}{k}_q\tau^k(1-\tau)_q^{n-k}\qquad 0<\tau<1\qquad k=0,1,\ldots,n. \tag{26}$$

When $q\to 1$, (26) approaches the usual binomial distribution. By virtue of the q-binomial theorem (23), we obtain the PGF for (26)

$$G_{\mathrm{B}}(n;z,\tau,q)=\sum_{k=0}^{n} z^kP_{\mathrm{B}}(k;n,\tau,q)=\sum_{k=0}^{n}\binom{n}{k}_q(\tau-\tau z)_q^k. \tag{27}$$

When $z=1$, (27) gives $G_{\mathrm{B}}(n;1,\tau,q)=1$, which means that the q-binomial distribution (26) is normalized. The limiting form of the PMF (26) can be derived easily,

$$\lim_{n\to\infty} P_{\mathrm{B}}(k;n,\tau,q)=\lim_{n\to\infty}\frac{[n]_q[n-1]_q\ldots[n-k+1]_q}{[k]_q!}\tau^k(1-\tau)_q^{n-k}$$

$$=\frac{(\tau/1-q)^k}{[k]_q!}(\mathrm{e}_q^{\tau/1-q})^{-1} \tag{28}$$

which is the Euler distribution exactly. This expression means that the usual Poisson's theorem about the limiting form of the binomial distribution is true in the q-deformed case. It may be called the q-deformed Poisson's theorem. Based on this theorem, it is easily seen that the limiting form of the PGF (27) is

$$\lim_{n\to\infty} G_{\mathrm{B}}(n;z,\tau,q)=\mathrm{e}_q^{z\tau/1-q}(\mathrm{e}_q^{\tau/1-q})^{-1} \tag{29}$$

which is the PGF of the Euler distribution as expected.

4. Properties and applications of the q-binomial distribution

The recurrence relationship for the three distributions given by (20), (24) and (26) are respectively:

$$\frac{P_{\mathrm{NB}}(k;n,\alpha,q)}{P_{\mathrm{NB}}(k-1;n,\alpha,q)}=\frac{1-q^{n+k-1}}{[k]_q}\frac{\alpha}{1-q}\qquad 0<q<1\qquad 0<\alpha<1 \tag{30}$$

$$\frac{P_{\mathrm{B}}^{*}(k;n,\beta,Q)}{P_{B}^{*}(k-1;n,\beta,Q)}=\frac{1-Q^{n-k+1}}{[k]_{1/q}}\frac{\beta}{1-q}\qquad 0<Q<1\qquad 0<\beta \tag{31}$$

$$\frac{P_B(k;n,\tau,q)}{P_B(k-1;n,\tau,q)}=\frac{1-q^{n-k+1}}{[k]_q}\frac{\tau}{1-q}(1-q^{n-k}\tau)^{-1} \qquad 0<q<1 \qquad 0<\tau<1. \quad (32)$$

Many interesting quantities such as the mean value and the mean square deviation can be determined from the PGF for a distribution. To do this, let us introduce the q-derivative defined by [4]

$$\frac{d}{d_q z}f(z)=\frac{f(z)-f(qz)}{(1-q)z}. \quad (33)$$

In the q-deformed binomial distribution, we have

$$\frac{d}{d_q z}G_B(n;z,\tau,q)|_{z=1}=\sum_{k=0}^{n}[k]_q P_B(k;n,\tau,q)=[n]_q\tau=E([k]_q) \quad (34)$$

$$\frac{d^2}{d_q z\, d_q z}G_B(n;z,\tau,q)|_{z=1}=\sum_{k=0}^{n}[k]_q[k-1]_q P_B(k;n,\tau,q)$$
$$=[n]_q[n-1]_q\tau^2=q^{-1}(E([k]_q^2)-E([k]_q)) \quad (35)$$

where $E([k]_q)$ is the mean value of $[k]_q$ in the q-deformed binomial distribution, $E([k]_q^2)$ the second moment of $[k]_q$, and the fact $[k-1]_q=q^{-1}([k]_q-1)$ has been used in (35). Equations (34) and (35) lead to the mean square deviation of $[k]_q$ in the q-deformed binomial distribution

$$D([k]_q)=E([k]_q^2)-(E([k]_q))^2=[n]_q\tau(1-\tau)$$
$$=\left(q\frac{d^2}{d_q z\, d_q z}G_B(n;z,\tau,q)+\frac{d}{d_q z}G_B(n;z,\tau,q)-\left(\frac{d}{d_q z}G_B(n;z,\tau,q)\right)^2\right)\Bigg|_{z=1}. \quad (36)$$

As an application of the q-deformed binomial distribution introduced in the previous section, we construct the binomial state for a single-mode of the q-boson as

$$|\tau,m\rangle=\sum_{n=0}^{m}\sqrt{P_B(n;m,\tau,q)}|n\rangle \quad (37)$$

where $|n\rangle$ is the q-number state. Because $G_B(n;1,\tau,q)=1$, the state $|\tau,\mathrm{m}\rangle$ is normalized to

$$\langle\tau,m|\tau,m\rangle=\sum_{n=0}^{m}P_B(n;m,\tau,q)=1. \quad (38)$$

It is also easily seen from the definition (37) that for $\tau=0$ and 1, with m finite, the q-binomial state $|\tau,m\rangle$ reduces to the vacuum state $|O\rangle$ and the q-number state $|n=m\rangle$ respectively. In the limit $m\to\infty$, the q-binomial state $|\tau,m\rangle$ will approach a q-coherent state $|\alpha=\sqrt{\lambda}\rangle$ because of (28), where the parameter $\lambda=\tau/(1-q)$. Therefore, the q-binomial state (37) possesses all the desired limiting behaviour as an interpolating state between the q-number and the q-coherent states and reduces to the ordinary binomial state when $q\to1$. Furthermore, using (34) and (36), we can calculate the mean value and the mean square deviation of the operator $[N]_q=a^+a$ for the state $|\tau,m\rangle$

$$\langle[N]_q\rangle=\langle\tau,m|[N]_q|\tau,m\rangle=[m]_q\tau \quad (39)$$

$$\langle([N]_q-\langle[N]_q\rangle)^2\rangle=\langle\tau,m|([N]_q-\langle[N]_q\rangle)^2|\tau,m\rangle=[m]_q\tau(1-\tau) \quad (40)$$

respectively. The ratio of deviation to mean (Fano factor) for the state $|\tau, m\rangle$ is

$$\frac{\langle([N]_q-\langle[N]_q\rangle)^2\rangle}{\langle[N]_q\rangle}=1-\tau<1 \tag{41}$$

which shows the sub-Poissonian nature of the q-binomial distribution.

The bunching parameter for the state $|\tau, m\rangle$ is

$$\langle([N]_q-\langle[N]_q\rangle)^2\rangle-\langle[N]_q\rangle=-[m]_q\tau^2<0. \tag{42}$$

The second-order correlation function for the state $|\tau, m\rangle$ is

$$\frac{\langle\tau, m|a^{+2}a^2|\tau, m\rangle}{\langle\tau, m|a^+a|\tau, m\rangle^2}=\frac{[m-1]_q}{[m]_q}=q^{-1}\left(1-\frac{1}{[m]_q}\right)<1 \qquad (m\geqslant 2). \tag{43}$$

Equations (42) and (43) indicate that the q-binomial states are antibunched. As stated in [12], antibunching and sub-Poissonian behaviour always accompany each other for single-mode time-independent fields; so do the q-binomial states.

In summary, in order to construct the binomial state for the q-boson, we define a new distribution (the q-binomial distribution) in this paper, which takes the Euler distribution as its limiting form. Its application to the q-boson theory is shown. We are currently trying to formulate other meaningful models for the q-deformed binomial distribution.

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References

[1] Glauber R J 1963 *Phys. Rev.* **130** 2529; 1963 *Phys. Rev.* **131** 2766
[2] Stoler D, Saleh B E A and Teich M C 1985 *Opt. Acta* **32** 345
[3] Biedenharn E C 1989 *J. Phys. A: Math. Gen.* **22** L873
Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Sun C-P and Fu H-C 1989 *J. Phys. A: Math. Gen.* **22** L983
[4] Chaturvedi S, Kapoor A K, Sanya R and Srinivasan V 1991 *Phys. Rev.* A **43** 4555
Chaturvedi S and Srinivasan V 1991 *Phys. Rev.* A **44** 8020; 8024
Chiu S H, Gray R W and Nelson C A 1992 *Phys. Lett.* **164A** 237
Ellinas D 1992 *Phys. Rev.* A **45** 3358
Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415
[5] Benkherouf L and Bather J A 1988 *J. Appl. Probability* **25** 529
[6] Kemp A W 1992 *Commun. Statist. Theory Meth.* **21** 571
Kemp A W 1992 *J. Appl. Probability* **29** 869
[7] Jackson F H 1910 *Quart. J. Math.* **41** 193
[8] Andrews G E 1985 *q*-series: their development and application *Conference Board of the Mathematical Science Pub.* **66**
[9] Dunkl C E 1981 *Commun. Statist. Theory. Meth.* A **10** 1915
[10] Kemp A W and Kemp C D 1991 *Am. Statistician* **45** 216
[11] Jing S-C and Fan H-Y 1993 *J. Phys. A: Math. Gen.* **26** L69
[12] Teich M C, Saleh B E A and Stoler D 1983 *Opt. Commun.* **46** 244